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**Key Points:**

- A simple, generalized derivation of the Dieterich (1994) seismicity rate theory is presented
- Simple integral expressions are derived for seismicity rate and cumulative number of events
- Many fundamental assumptions are analyzed and their validity tested

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# Constitutive Law for Earthquake Production Based on Rate-and-State Friction: Dieterich 1994 Revisited

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**Abstract** Dieterich (1994, https://doi.org/10.1029/93JB02581) derived a constitutive law for earthquake production based on rate-and-state friction, which has since been applied widely to earthquake triggering in various tectonic settings. Here this influential work is revisited and rederived in a more straightforward manner. Our derivation is based on computing the time to instability for a population of sources, and eschews the seismicity state variable. We demonstrate the validity of the Dieterich (1994) model for arbitrary shear stressing history at constant normal stress; however, our results differ slightly if fault normal stress changes with time. We provide simple integral expressions for the cumulative number of events and seismicity rate for arbitrary stressing history. These expressions have no explicit dependence on the time derivative of the stressing history. The simpler derivation makes it easier to assess and generalize various model assumptions in the original formulation. A principal success of the Dieterich (1994) theory is that it predicts and gives physical meaning to the Omori decay of seismicity rate following a stress step. We analyze the assumption that sources are well above steady state and find that Omori decay is produced by sources that either start or end up above steady state following the stress step. If no sources are brought above steady state by the perturbation then the time to reach steady state must be considered, and there will be no Omori sequence.

## 1. Introduction

In the seminal work of Dieterich (1994; hereinafter referred to as Dieterich's theory or model), James Dieterich bridged a gap between the fields of statistical seismology and fault mechanics. He proposed a constitutive relationship between the current rate of seismicity, $R$, and the stressing history, based on rate-and-state friction and certain assumptions detailed below. Specifically, he posited

$$R = \frac{r}{\dot{\tau}_r \gamma}, \tag{1}$$

where $r$ the background rate of seismicity, $\dot{\tau}_r$ is the background shear stressing rate and $\gamma$ is a state variable that depends on the shear and normal stressing history acting on the population of potential seismic sources. Dieterich derived the following ordinary differential equation (ODE) for $\gamma$

$$\dot{\gamma} = \frac{1}{A\sigma}\left[1 - \gamma\dot{\tau} + \gamma\left(\frac{\tau}{\sigma} - \alpha\right)\dot{\sigma}\right], \tag{2}$$

where $A$ is a constitutive parameter relating changes in instantaneous slip rate to friction, $\sigma$ and $\tau$ are the effective normal stress and shear stress (respectively) acting on the population of sources, and $\alpha$ is a constitutive parameter that relates change in normal stress to friction (Linker & Dieterich, 1992).

The model assumes that the crust contains a populations of seismogenic sources subject to spatially uniform initial and perturbing stress fields. Furthermore, it assumes that each population produces a constant seismicity rate $r$ under constant shear stressing rate $\dot{\tau}_r$, and constant normal stress ($\dot{\sigma} = 0$), which is seen by noting that the steady-state solution to (2) is $\gamma = 1/\dot{\tau}_r$.

Dieterich (1994) showed that for a step change in stress equations (1)–(2) predict the well-known Omori's law of aftershocks with the characteristic 1/time decay of earthquake rate and thus gives physical meaning to the empirical law. Although Dieterich argues that (2) holds for arbitrary stressing history, he only explicitly demonstrates this to be true for a few simple stressing histories, none of which include time-varying normal stress. In this study we rederive Dieterich's constitutive relationship and explicitly show that equations (1)–(2) are true for arbitrary shear stressing history; however, our derivation leads to a slightly modified expression when normal stress is time-varying. This work mathematically validates the model for complicated stressing histories, although more direct comparisons with observations of earthquake triggering from temporally complicated stressing are needed to more generally establish the validity of the model. Furthermore, we show that the state variable $\gamma$ can be eliminated, as well as the ODE (2), and the seismicity rate $R$ cast in the form of a single definite integral.

Dieterich's theory has been widely used to model seismicity sequences, due to its mathematical simplicity and agreement with Omori's law. Most applications have considered simple stressing, such as a step change in shear stress due to a large earthquake. Some studies have looked at more complicated stressing histories and shown the model to be consistent with observations or independent data sets such as geodetic data (e.g., Dieterich et al., 2000; Green et al., 2015; Inbal et al., 2017; Segall et al., 2013; Ziv, 2012). However, as was previously mentioned, there is still need for further verification. In particular, in the presence of large changes in effective normal stress such as for fluid injection or production induced seismicity, where stresses evolve over a relatively long timespans, and in volcanic regions.

A few studies have extended the model. Dieterich et al. (2000) showed how to extract changes in stress directly from observed seismicity. This was also investigated by Helmstetter and Shaw (2009) who found an analytical expression that maps seismicity rate and cumulative number of events to stress change. A number of observational studies have utilized this result (e.g., Inbal et al., 2017; Savage, 2010) since seismicity rate and cumulative number of events can be estimated directly from data. Spatially heterogeneous stresses were explored by Helmstetter and Shaw (2006), who related the $p$ value of Omori's law to stress heterogeneity. The model as originally formulated, ignores stress interactions between sources within a population. However, Ziv and Rubin (2003) showed that for coplanar sources where the net contribution of each source to the effective stressing rate is approximately constant and the earthquake magnitude distribution is not altered by the stress perturbation, then the prediction of Omori decay following a stress step is recovered in spite of the interactions. To the best of our knowledge, no previous study has rederived the Dieterich model for arbitrary stressing history; however, Gomberg et al. (2005) demonstrated an alternative way of constructing the Dieterich prediction of Omori's law and provided insight into the relationship between the clock advance of sources and their maturity.

A common approximation in the literature assumes the ratio of shear to normal stress ($\tau/\sigma$) and the factor $1/A\sigma$ to be constant in equation (2). This allows the evolution of stress to be couched in terms of Coulomb stress change, which we refer to as the Coulomb stress approximation. We show that this does not converge to the correct solution when the normal stress is not constant. In numerical implementations the ODE (2) is often integrated iteratively assuming assuming piecewise constant Coulomb stress change (e.g., Catalli et al., 2008; Green et al., 2015; Hainzl et al., 2010). More efficient numerical methods have also been developed, for example, making use of analytical solutions for piecewise constant Coulomb stressing rate (Cattania & Khalid, 2016; Segall et al., 2013). However, this approach will become unstable if stressing rate is zero and may be subject to inaccuracies for very rapid changes in stress. Here we derive general expressions for the seismicity rate and cumulative number of events ($N$) in terms of an integral of the temporally varying stresses, which can be solved more efficiently and accurately than the time stepping procedure to compute $R$. In many cases $R$ is integrated numerically to obtain $N$ (e.g., Catalli et al., 2008; Hainzl et al., 2010); however, we provide an expression for $N$ in terms of the same integral. The analytical expressions derived here give more insight into the relationship between stress and seismicity rate or cumulative number of events than previous formulations.

In section 2, we derive an expression for the time to instability of a spring-slider source subject to arbitrary stress history, assuming that friction is initially well above steady state. This result is then extended to describe a population of sources. From there, we retrieve integral expressions that are nearly equivalent to the Dieterich model, however, differ slightly when normal stress is time dependent. We refer to this as the *modified Dieterich model*. The model is next generalized to investigate cases where the background rate of seismicity is not constant and when the background stressing rate includes normal and shear components. Furthermore, we analyze the Coulomb stress approximation and investigate the validity of assuming that sources are well above steady state. This paper aims to provide explicit and straightforward derivations, such that not only can the reader utilize the expressions but also extend the model in useful ways not considered here.

## 2. Theory

### 2.1. The Time to Instability

In this section we formulate an approximate expression for time to instability for a simple spring-slider model. The spring-slider can be thought of as a simple model of a patch on a fault, or a fault segment. In this section we focus only on a single spring-slider source, but in the following sections we apply the results to a population of sources.

Equating the shear stress and the frictional resistance, the latter the product of the effective normal stress $\sigma$ and coefficient of friction, where friction depends on instantaneous slip speed $v = \dot{\delta}$, ($\delta$ is displacement) and state variable, $\theta$,

$$\tau(t) - k\delta(t) = \sigma\left(\mu + A\log\frac{\dot{\delta}(t)}{V^*} + B\log\frac{\theta(t)V^*}{d_c}\right). \tag{3}$$

Here $\tau$ is the applied shear stress, $-k\delta$ the shear stress relaxed by slip, and $k$ the effective stiffness of the source patch. $A$ and $B$ are constitutive parameters relating friction to changes in slip speed and state, respectively, $V^*$ is a reference value of the slip speed, and $d_c$ is a characteristic slip distance over which state evolves. Finally, $\mu$ is a reference coefficient of friction such that if $\dot{\delta} = V^*$ and $\theta = d_c/V^*$, then $(\tau - k\delta)/\sigma = \mu$. Note that the momentum balance in (3) ignores inertial terms and thus only holds when slip speeds are small enough that elastodynamic effects are negligible.

We assume, following Dieterich (1994), that state evolves according to the aging law (Ruina, 1983), where we have included, for completeness, the correction for time-dependent normal stress (Linker & Dieterich, 1992):

$$\dot{\theta} = 1 - \frac{\dot{\delta}\theta}{d_c} - \frac{\alpha\theta}{B\sigma}\dot{\sigma}, \tag{4}$$

where $\alpha$ is a constitutive parameter relating change in normal stress to the change in state, where $0 \leq \alpha \leq \mu$. Equation (4) is a first order linear ODE with a time-dependent coefficient and is, therefore, integrable (e.g., Polyanin & Zaitsev, 1995):

$$\theta(t) = e^{F(t)}(\theta_0 + \int_0^t e^{-F(t')}\mathrm{d}t'), \tag{5}$$

where

$$F(t) = -\left[\frac{\delta(t)}{d_c} + \frac{\alpha}{B}\log\left(\frac{\sigma(t)}{\sigma_0}\right)\right]. \tag{6}$$

Well above steady state, the so called *no-healing limit*, that is, omitting the 1 on the right-hand side of (4), is a good approximation (Dieterich, 1992). We return to the validity of this approximation in section 3.1. In the no-healing limit, (5) reduces to

$$\theta(t) = \theta_0 e^{F(t)} \tag{7}$$

Substituting $\theta(t)$ from (7) and (6) into (3) yields

$$\frac{\tau(t) - k\delta(t)}{\sigma(t)} = \mu + A\log(\dot{\delta}/V^*) - B\left[\frac{\delta}{d_c} + \frac{\alpha}{B}\log\left(\frac{\sigma(t)}{\sigma_0}\right)\right] + B\log\left(\frac{V^*}{d_c}\theta_0\right). \tag{8}$$

Equation (8) is a first-order ODE for $\dot{\delta} \equiv d\delta/dt$, which can be rearranged as

$$K(t)dt = \frac{1}{\dot{\delta}_0}\exp\left(\left[\frac{k}{A\sigma} - \frac{B}{Ad_c}\right]\delta\right)d\delta, \tag{9}$$

where $\dot{\delta}_0 = V^*\exp(\tau_0/A\sigma_0 - \mu/A)(V^*\theta_0/d_c)^{-B/A}$ is the slip speed at time $t = 0$ and $K$ is an integral kernel:

$$K(t) = \exp\left(\frac{\tau(t)}{A\sigma(t)} - \frac{\tau_0}{A\sigma_0}\right)\left(\frac{\sigma(t)}{\sigma_0}\right)^{\alpha/A}. \tag{10}$$

Equation (9) can be integrated to solve for $\delta$ only when $k/\sigma$ is independent of time. Scaling arguments show that for slipping zones in elastic continua $k$ scales as $G/l$, where $G$ is shear modulus and $l$ is the half length of the nucleation patch (e.g., Dieterich, 1994). Specifically, $l$ scales with $l \sim Gd_c/\phi\sigma$, where $\phi$ depends on the constitutive parameters $A$ and $B$. For $A/B \lessapprox 0.38$, Rubin and Ampuero (2005) showed that for flat faults with heterogeneous stress, nucleation occurs on a patch with fixed dimension where $\phi \approx B/1.38$ (see also Viesca, 2016). For larger $A/B$, Rubin and Ampuero (2005) showed that the accelerating nucleation zone approaches a limiting dimension, where $\phi \approx \pi(B-A)^2/B$. In this case, $k$ is inherently time dependent as it approaches the limiting value. On the other hand, it is possible that the nucleation dimension is constrained by geometric or strong stress heterogeneity, so as to still scale inversely with normal stress, but not grow as much as predicted by the Rubin and Ampuero (2005). In these cases it is sensible to follow Dieterich (1994) and approximate

$$\left[\frac{k}{A\sigma} - \frac{B}{Ad_c}\right] \approx \left[\frac{k}{A\sigma_0} - \frac{B}{Ad_c}\right] = -H/A \quad \text{where} \quad H = \frac{B}{d_c} - \frac{k}{\sigma_0} \approx \frac{B-\phi}{d_c}. \tag{11}$$

Even if the stiffness $k$ is independent of normal stress, but normal stress varies in time, then a Taylor expansion of the exponent in the normal stress perturbation $\Delta\sigma(t)$ suggest that if $|\Delta\sigma(t)/\sigma_0| \ll 1$ then (11) is a good approximation. It is worth noting that Kaneko and Lapusta (2008) compared predictions of continuum models with a weak fault patch to spring slider models and found that the two were in general agreement, in particular for constant normal stress in the nucleation region. Their findings support the assumption of considering $H$ time invariant. This simplification allows for an analytical solution for the slip and slip speed as a function of time:

$$\delta = -\frac{A}{H}\log\left[1 - \frac{H\dot{\delta}_0}{A}\int_0^t K(t')dt'\right], \tag{12}$$

$$\dot{\delta} = \frac{\dot{\delta}_0 K(t)}{1 - \frac{H\dot{\delta}_0}{A}\int_0^t K(t')dt'}. \tag{13}$$

Because we have ignored inertial effects, the slip speed becomes singular when the denominator in the above expression vanishes. We take this to define the time to instability $t_{inst}$,

$$\int_0^{t_{inst}} K(t')\mathrm{d}t' = \frac{A}{H\dot{\delta}_0}. \tag{14}$$

Note that the time to instability depends on the stressing history through the kernel $K(t)$ and the initial conditions through $\dot{\delta}_0$.

It is useful to review the approximations made in this section. We assume that the nucleation of a single source can be represented by a spring-slider model with constant stiffness. Secondly, we assumed that the state evolution can be well approximated by the aging law well above steady-state (the no-healing approximation). Finally, we assumed that the quantity $H$ in (11) is time invariant or can be approximated as such due to nucleation dimension scaling inversely with the normal stress, which allows (9) to be integrated.

### 2.2. Populations of Sources

For constant normal stress and shear stressing rate, $\tau(t) = \tau_0 + \dot{\tau}_r t$, one can show from equation (14) that the time to instability is

$$t_{inst} = \frac{A\sigma_0}{\dot{\tau}_r}\log\left(\frac{\dot{\tau}_r}{H\dot{\delta}_0\sigma_0} + 1\right), \tag{15}$$

(equation (A13) in Dieterich, 1994). We now consider a population of sources that fail under background conditions at constant rate $r$. The background conditions are taken to be $\tau(t) = \tau_0 + \dot{\tau}_r t$ and $\sigma(t) = \sigma_0$. Since failures occur at a constant rate, the time to instability of the $N$th source is $t = N/r$. We drop the subscript $_{inst}$, when referring to a population of sources, since time to instability can represent continuous time if $N$ is allowed to take noninteger values. Thus, equation (14) for background conditions becomes

$$\int_0^{N/r} \exp(\dot{\tau}_r t'/A\sigma_0)\mathrm{d}t' = \left(\frac{A}{H\dot{\delta}_0}\right)_N. \tag{16}$$

The right-hand sides of equations (14) and (16), $A/H\dot{\delta}_0$, is the time to instability at constant stress and is fully determined by the initial conditions. Equation (16), therefore, gives the value $A/H\dot{\delta}_0$ for the $N$th source in a population of sources that fail at constant rate $r$ under background conditions.

We can now take the distribution defined by equation (16) and substitute for $A/H\dot{\delta}_0$ in (14)

$$\int_0^t K(t')\mathrm{d}t' = \int_0^{N/r} \exp(\dot{\tau}_r t'/A\sigma_0)\mathrm{d}t'. \tag{17}$$

Equation (17) relates the time to instability $t(N)$ of the $N$th source for arbitrary stressing history in a population of sources that would fail at constant rate $r$ under background conditions. Solving for $N$ yields the cumulative number of nucleated earthquakes

$$\frac{N}{r} = t_a \log\left(\frac{1}{t_a}\int_0^t K(t')dt' + 1\right), \tag{18}$$

where we have made use of Dieterich's definition of the *characteristic aftershock decay time*, $t_a$,

$$t_a = \frac{A\sigma_0}{\dot{\tau}_r}. \tag{19}$$

The expression (18) is useful when modeling cumulative seismicity or inverting the number of earthquakes, as it is easy to solve analytically in many cases. When numerical methods are needed only a single integral has to be evaluated. Note that (18) does not include explicit dependence on the rate of change of stress, with the exception of the background stressing rate. This makes the formula simpler than (1) and (2) where stressing rates might have to be computed using finite differences.

It is worth noting that (18) is more general than Dieterich's theory when the perturbing normal stress is nonzero, since the initial shear stress need not be constant for all sources in the population. If initial stress depends on source number, $\tau_0(N(t))$, the kernel $K$ depends on both the integral time variable $t'$ and time to instability $t$, that is, $K(t', N(t))$. Clearly, in this case computing $N(t)$ is more complicated then when $\tau_0$ is constant. For constant normal stress there is no complication because the initial shear stress cancels in $K$ (equation (10)). In section 4.2 we argue that we may usually consider $\tau_0$ as constant when it occurs in $K$, which we will assume throughout this paper.

Differentiating (18) with respect to $t$ to obtain $R = dN/dt$, and assuming the $\tau_0$ is independent of source number or that normal stress is constant in time (as in the previous paragraph), the seismicity rate is

$$\frac{R}{r} = \frac{K(t)}{1 + \frac{1}{t_a}\int_0^t K(t')dt'} = \frac{K(t)}{1 + \frac{\dot{\tau}_r}{A\sigma_0}\int_0^t K(t')dt'}. \tag{20}$$

It is easy to verify that for a shear stress step, $K = \exp(\Delta\tau/A\sigma_0 + t/t_a)$, then (20) reduces to

$$\frac{R}{r} = \left[\left(e^{-\Delta\tau/A\sigma_0} - 1\right)e^{-t/t_a} + 1\right]^{-1}, \tag{21}$$

which agrees with Dieterich (1994, equation (12)) in the limit that the prestep and poststep stressing rates are the same. For times short compared to the aftershock decay time $t/t_a \ll 1$, (21) has the Omori-Utsu form $R = a/(b+t)$.

In Appendix A we integrate the ODE (2) for $\gamma$. Comparison of equations (20) and (A7) shows that if $\sigma = \sigma_0$ the two approaches are the same. Thus, Dieterich's model (equations (1) and (2)) is generally true if the normal stress is constant. However, if normal stress is not constant, Dieterich's expression is slightly different since $\sigma_0$ in the denominator of (20) is replaced by $\sigma(t)$ in (A7). In the derivation here, $\sigma_0$ arises from the background population of sources, which is assumed to be at constant normal stress (equation (16)), as is true in Dieterich's formulation. The difference thus does not stem from this assumption. Savage (2010, Appendix A) independently found an expression equivalent to (A7) by integrating equation (2) for the special case when normal stress is constant.

Equations (18) and (20) are different in several ways from the fundamental results of Dieterich (1994) (equations (1) and (2)). They differ in derivation, they present semianalytical expressions for both cumulative

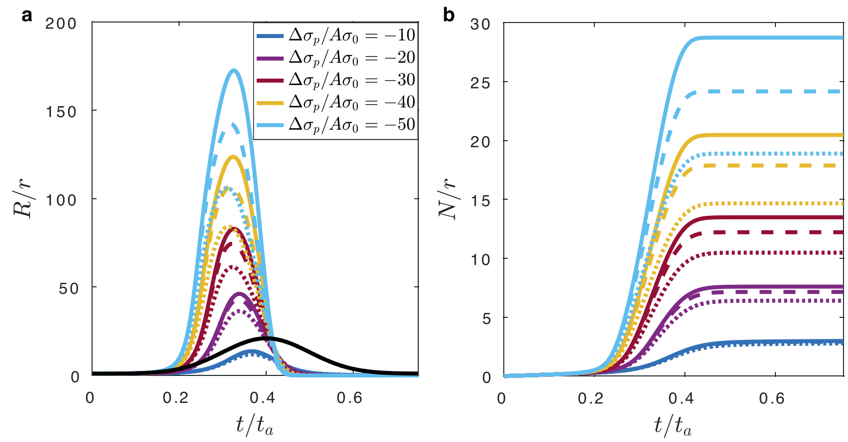


**Figure 1.** (a) Comparison of seismicity rate for modified Dieterich theory equation (20) (solid line), original theory equation (A7) (dashed line), and Coulomb stress approximation (dotted line). Black line indicates the shape of the Gaussian normal stress perturbation, and $\Delta\sigma_p$ is the peak stress of the Gaussian perturbation. (b) Same as for (a), but showing the cumulative number of events.

number of events and the seismicity rate, and finally, they differ in their dependence of normal stress, which can become significant for large changes in normal stress. In this paper we may refer to these results as the "modified Dieterich model," to distinguish from the original results of Dieterich (1994), such as equations (1) and (2), if such distinction is needed. The exact origin of the difference is not clear since we make no explicit assumptions that differ from the assumptions made by Dieterich (1994). Determining which model better agrees with observations could be challenging since they only deviate significantly for very large changes in normal stress as shown in Figure 1.

The differences in the modified Dieterich theory and the original theory, for example, equation (A7), become evident when normal stress changes are large, that is also when the Coulomb stress approximation (addressed in section 3.2.2) becomes poor. To demonstrate this difference, we compute the seismicity rate and cumulative number of events for slowly varying normal stress that is Gaussian in time with variable peak stress $\Delta\sigma_p$ (Figure 1). This may be comparable to conditions the lead to injection-induced seismicity in the near field of an injector where effective stress perturbations are large. The results clearly show that there are considerable differences in seismicity rate and cumulative number of events in the case of large normal stress changes. Making use of the approximate kernel $K$ in equation (29), which results in the popular Coulomb stress approximation discussed later results in further disagreement, which should be considered when applying that approximation to injection-induced seismicity.

It is useful to review the approximations made in this section. We first assumed that the background stressing rate consists of a constant shear stressing rate, $\dot{\tau}_r$, and constant normal stress $\sigma_0$. To derive the seismicity rate from (18), we further assumed that either the initial shear stress $\tau_0$ is the same for all sources or that normal stress is invariant in time.

## 3. Addressing the Assumptions

In this section we analyze several assumptions that were made in the derivation of the Dieterich model or are commonly made in applications of the model. We provide possible ways to relax these assumptions or explain why they are reasonable.

### 3.1. Well Above Steady State: Why does it work?

One of the puzzles of the Dieterich model is that the derivation explicitly assumes that seismogenic sources are well above steady state (section 2), yet they must spend most of the seismic cycle well below steady state. Why does it work? To answer this we must first consider what it means "to work." The success of the model is primarily that it predicts Omori law behavior following a rapid change in shear stress. Another feature is that the model is consistent with a steady-seismicity rate given a constant rate of stressing, although to be fair the model is constructed to produce that behavior. We consider this aspect first.

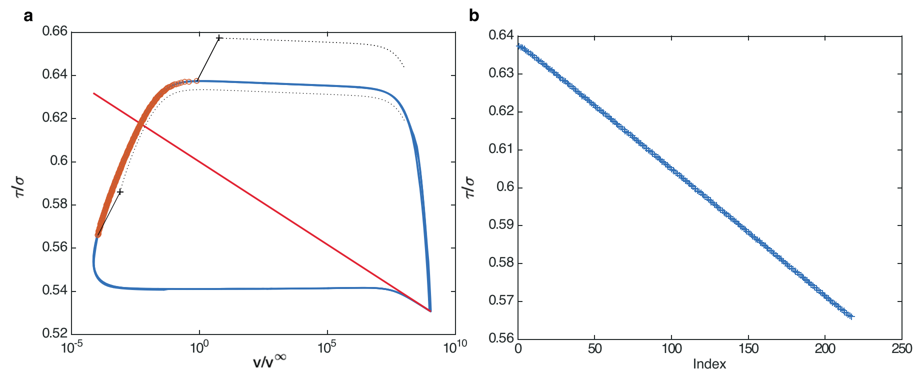


**Figure 2.** (a) Trajectories in normalized stress-log velocity space. Steady-state line (red), stick-slip cycles (blue), and initial states spaced at 1 year intervals prior to instability (orange circles). The unperturbed cycle time is roughly 290 years. Following the stress step, $\Delta\tau/a\sigma = 2$, the first and last sites are displaced to the plus signs; thereafter, they follow the dotted trajectories. For this simulation $a/b = 0.75, a = 0.01$ and the spring stiffness is 0.1 of the critical stiffness. (b) $\tau_N/\sigma$ for $N$ sources spaced 1 year apart in terms of time to failure under background conditions.

From equations (15) and (19) the time to instability of the $N$th source $N/r$ is

$$\frac{N}{r} = t_a \log\left[\frac{A}{t_a H\left(\dot{\delta}_0\right)_N} + 1\right], \tag{22}$$

where $(\dot{\delta}_0)_N$ represents the initial slip speed of the $N$th source. Solving for the distribution of initial velocities that produce the constant rate of seismicity $r$, and substituting into the friction law (the RHS of (3) yields

$$\frac{\tau_N}{\sigma} - \mu = B\log\left(\frac{(\theta_0)_N V^*}{d_c}\right) + A\log\left(\frac{A}{V^* H t_a}\right) - A\log\left(e^{N/rt_a} - 1\right), \tag{23}$$

where $\tau_N$ and $(\theta_0)_N$ are the shear stress and state of the N-th source. Pairs of stress and state, initially above steady state, that satisfy (23) for $N = 1, 2, \ldots$ will fail at constant rate. Well below steady-state stress increases at constant rate $\dot{\tau}_r$, while $\dot{\theta} \simeq 1$, such that friction increases (ages) weakly with time. Thus, sources that satisfy (23) but start below steady state will not generally reach steady state at a constant rate. However, for $N/rt_a > 1$ the last term in (23) is approximately $-AN/rt_a$. If $\theta_0$ is the same for all sources, as would be the case if they represent patches on a fault that last slipped dynamically in the same event, and/or $B$ is sufficiently small, then the difference in stress between sources is nearly constant (see Figure 2b). Thus, modulo the weak healing effect, sources *that start below steady state* with the same initial state, with equal stress increments and subject to a constant stressing rate will reach steady state, and hence fail, at a constant rate.

How long does it take a source to reach steady-state? Consider sources that start below steady state with state $\theta_{min}$ and shear stress $\tau_{min}$. Well below steady-state $\dot{\theta} \simeq 1$, so that at a later time the state is $\theta_{min} + t$. Thus, the time to reach steady state, $t_{ss}$, is

$$\tau(t_{ss}) = \sigma_0\left[\mu + (B - A)\log\left((\theta_{min} + t_{ss})V^*/d_c\right)\right]. \tag{24}$$

This represents a transcendental equation for $t_{ss}$, given a pair $(\tau_{min}, \theta_{min})$. Ignoring the healing effect, the RHS is independent of $t_{ss}$, and at constant stressing rate the LHS is simply $\dot{\tau}_r t_{ss}$. This leads to a simple approximation, which we label $\hat{t}_{ss}$, which can be improved by the first-order correction for healing, $\sigma_0(B - A)\log\left[(\theta_{min} + \hat{t}_{ss})/\theta_{min}\right]/\dot{\tau}_r$. Now turn to the Omori decay prediction. Taking the approximation for $H$ in (11), (22) can be written as

$$t_{inst} = t_a\log\left[\left(\frac{A}{B - \phi}\right)\frac{d_c}{\dot{\delta}_0 t_a} + 1\right] = t_a\log\left[\left(\frac{\phi}{B - \phi}\right)\frac{v^\infty}{\dot{\delta}_0} + 1\right]. \tag{25}$$

The second form takes advantage of the spring slider equation $\dot{\tau} = k(v^\infty - \dot{\delta})$, where $v^\infty$ is the load-point velocity. Thus, the remote stressing rate is $\dot{\tau}_r = kv^\infty$. Here $\dot{\delta}_0$ is the initial slip speed *for a source above steady state*. There are two limiting behaviors: For $d_c/\dot{\delta}_0 \ll t_a$ then $t_{inst} \sim d_c/\dot{\delta}_0$ (assuming $A/(B - \phi)$ is of order one). On the other hand for $d_c/\dot{\delta}_0 \gg t_a$ or $v^\infty \gg \dot{\delta}_0$ then $t_{inst} \sim Ct_a$, where $C$ is a constant of order 1.

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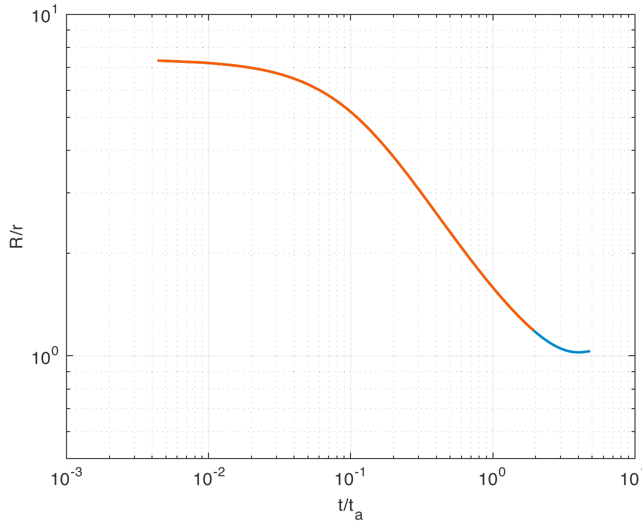


**Figure 3.** Normalized seismicity rate $R/r$ following a stress step, $\Delta\tau/a\sigma = 2$, as a function of time since the perturbation $t/t_a$. Blue are sources that are *not* brought above steady state by the stress step.

At the time of a step in shear stress some sources will already be above steady state. Others start below steady state but are pushed above by the stress step, which increases the slip speed by a factor of $\exp(\Delta\tau/A\sigma_0)$. Those sources that end up at the highest velocity will fail at times or order $t_{inst} \sim d_c/\dot{\delta}_0$, where $\dot{\delta}_0$ is the slip speed following the stress perturbation. Those at the lowest velocity, but above steady state will fail at time $t_{inst} \sim t_a$. Thus, what controls the aftershock decay time are the last sources that are elevated above steady state by the stress perturbation. Sources that remain below steady state must be brought to steady state by background stressing, and thus reach steady state (and hence fail) at a constant background rate—They are not part of the aftershock process. It follows that if no sources are brought above steady state by the stress step, there will be no distinct aftershock sequence.

We test these predictions with numerical simulations of the spring-slider system. The procedure is as follows: For given friction parameters run simulations with constant load-point velocity until a stable limit cycle is reached (Figure 2a). From the time of the last instability pick $\tau_N, \theta_N, \left(\dot{\delta}_0\right)_N$ at times $\Delta t_N = N/r$ prior to instability—These initial conditions are thus guaranteed to produce a steady rate of failures subject to constant stressing rate. Note that some sources are above steady state, while the majority are not. Figure 2b shows that the initial stress decreases nearly linearly with time to instability (Index). Next, perturb these initial conditions by a stress step such that the final shear stress is $\tau_N + \Delta\tau$, increasing the slip speed by a factor of $\exp(\Delta\tau/A\sigma)$. Compute the times to failure given these initial conditions and the background loading rate, and finally compute the normalized seismicity rate $dN/dt$ .

The result in Figure 3 shows that, as expected, sources brought above steady state by the stress perturbation fail at times $t \lessapprox 2t_a$, whereas sources that were left below steady state after the perturbation generate seismicity at a rate near the background rate. The same behavior is observed with different initial conditions, including those at uniform initial shear stress, but varying initial state. Our conclusion is that Omori-like behavior is generated by sources that either start or end up above steady state following the stress step, where the time to instability is well approximate by the no-healing result (15).

The Dieterich (1994) model for a stress step (21) predicts a nearly constant rate of seismicity for times $t \lessapprox t_e = t_a \exp(-\Delta\tau/A\sigma)$. We associate the nearly constant seismicity rate with sources that are on the nearly constant stress part of the orbit in phase space (Figure 2a), that is near peak stress. Recall from the discussion after (16) that the time to instability for sources at constant stress is $A/H\dot{\delta}_0$; thus, the seismicity rate is proportional to $\dot{\delta}_0$. The stress step increases $\dot{\delta}_0$, and hence $R$, by a factor of $\exp(\Delta\tau/A\sigma)$, as in equation (21). How long does this period last? Consider a source that would have failed at time $t_a$; thus, $A/H\dot{\delta}_0 \sim t_a$. After the stress step, it fails at time $t \sim A[H\dot{\delta}_0 \exp(\Delta\tau/A\sigma)]^{-1} = t_a \exp(-\Delta\tau/A\sigma)$, which agrees with Dieterich's definition of $t_e$. Thus, if one could observe this plateau in the early aftershock sequence it would indicate sources that were near peak stress, and already accelerating to instability at the time of the stress step.

### 3.2. Approximations in Terms of Coulomb Stress

In this section we discuss two approximations that are frequently made to the Dieterich model for cases in which the fault normal stress is not constant in time. The first is the assumption that the background shear stressing rate $\dot{\tau}_r$ can be replaced by a Coulomb stressing rate of the form $\dot{s}_r = \dot{\tau}_r - \mu\dot{\sigma}_r$. The second approximation is that the stressing history can be written in terms of Coulomb stress change $\Delta S(t) = \Delta\tau - \mu\Delta\sigma$. Approximating the stressing history in terms of Coulomb stress change was introduced by Dieterich et al. (2000), where it was noted that $\mu = \mu' - \alpha$, where $\mu'$ is a constant coefficient of friction. Dieterich et al. (2000) called $\Delta S(t)$ the modified stress function. The definition of the Coulomb stress here deviates from the classical definition, which can be referred to as unmodified Coulomb stress. However, the uncertainty in $\mu'$ can be roughly of the same order as $\alpha$ and thus the distinction between unmodified and modified Coulomb stress becomes unclear in practice. We will, therefore, refer to approximating the stressing history in terms of Coulomb stress change (either modified or unmodified) as *the Coulomb stress approximation*. We show that that indeed the definition of $\mu$ by Dieterich et al. (2000) gives the most accurate solution with $\mu' = \tau_0/\sigma_0$; however, in practice $\mu$ needs to selected carefully, as an incorrect value can give rise to significant errors.

#### 3.2.1. Coulomb Background Stressing Rate

Faults may not exclusively be subject to constant shear stressing rate under background conditions, but rather a combination of shear and normal loading. For example, normal faults subject to horizontal principal stress, jogs and bends along strike-slip faults, or where there are gradual changes in pore pressure. Several studies have applied Dieterich's model simply replacing $\dot{\tau}_r$ with a (modified) Coulomb stressing rate $\dot{s}_r = \dot{\tau}_r - \mu\dot{\sigma}_r$ (e.g., Dieterich et al., 2000; Segall et al., 2013). However, if we do the same for equation (15) we will not attain a solution to (14), even if $\alpha = 0$. By applying the method in section 2.2 and equating (14) for a general stressing history to the background conditions (and eliminating $A/H\dot{\delta}_0$), we find that for a distribution of sources with background rate $r$ but subject to nonzero background normal stressing rate

$$\int_0^t K(t')dt' = \int_0^{N/r} \exp\left(\frac{\tau_0 + \dot{\tau}_r t'}{A(\sigma_0 + \dot{\sigma}_r t')} - \frac{\tau_0}{A\sigma_0}\right)\left(\frac{\sigma_0 + \dot{\sigma}_r t'}{\sigma_0}\right)^{\alpha/A} dt'. \tag{26}$$

The right-hand side can be integrated analytically if $\alpha = 0$; however, one cannot generally solve for $N$ so that is of limited value. Differentiating both sides with respect to time renders a first-order nonlinear ODE:

$$\frac{R}{r} = K(t)\exp\left(\frac{\tau_0}{A\sigma_0} - \frac{\tau_0 + \dot{\tau}_r N/r}{A(\sigma_0 + \dot{\sigma}_r N/r)}\right)\left(\frac{\sigma_0 + \dot{\sigma}_r N/r}{\sigma_0}\right)^{-\alpha/A}. \tag{27}$$

We can see how the assumption in which $\dot{\tau}_r$ is replaced by a constant Coulomb stressing rate $\dot{s}_r$ is an approximation to equation (26) or (27). Bringing the normal stress-dependent factor (corresponding to the Linker and Dieterich, 1992, effect) into the exponential on the right-hand side and approximating the argument by a first-order Taylor expansion with respect to $\dot{\sigma}_r t$, (26) reduces to

$$\begin{aligned} \int_0^t K(t')dt' &= \int_0^{N/r} \exp\left(\frac{\dot{\tau}_r - (\tau_0/\sigma_0 - \alpha)\dot{\sigma}_r}{A\sigma_0} t'\right) dt' \\ \Rightarrow \qquad \frac{N}{r} &= \frac{A\sigma_0}{\dot{s}_r}\log\left(\frac{\dot{s}_r}{A\sigma_0}\int_0^t K(t')dt' + 1\right), \end{aligned} \tag{28}$$

which is equivalent to equation (18) if $\dot{\tau}_r$ is interchanged for $\dot{s}_r = \dot{\tau}_r - (\tau_0/\sigma_0 - \alpha)\dot{\sigma}_r$. This suggests that if the Coulomb stressing rate is used for the background stressing rate the coefficient of friction should be set to $\mu = \tau_0/\sigma_0 - \alpha$. Simulations (Figure 4) show that this choice indeed minimizes the error relative to equation (26) or (27). In contrast, choosing a value for $\mu$ based on a typical value for a coefficient of friction, for example, $\mu = 0.6$ (the yellow line for this choice of parameters), could result in error of the order of 10% for both $R$ and $N$ in this case. In the calculations shown in Figure 4, the integral of the kernel $K(t)$ in (28), being simply a stress step, was computed exactly. Finally, it should be noted that the Coulomb approximation tends to be better than simply ignoring the normal stressing rate, although knowledge of the ratio $\tau_0/\sigma_0$ may be limited.

#### 3.2.2. Coulomb Stress Change

We now apply the Coulomb approximation to the arbitrary stressing history kernel $K(t)$, through a first-order Taylor expansion in the normal stress perturbation:

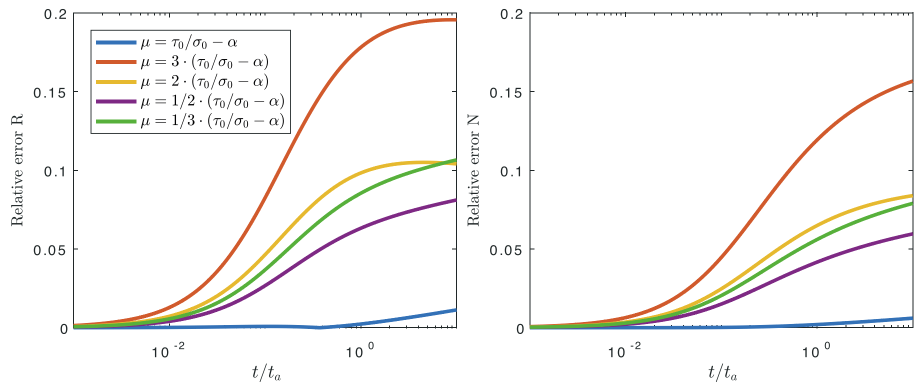


**Figure 4.** Comparison of the relative error in the seismicity rate and cumulative number of earthquakes using the Coulomb stressing approximation in the background stressing rate. If the cumulative number of events predicted by (26) is $N_{true}$, while (28) gives the approximation $N_{est}$, then the relative error is $|N_{true} - N_{est}|/N_{true}$. A stress step $\Delta\tau = 0.25$ MPa and $\Delta\sigma = -0.1$ MPa occurs at $t = 0$; the ratio $\dot{\tau}_r/\dot{\sigma}_r = -2$, with $\dot{\sigma}_r < 0$. Here $\tau_0/\sigma_0 - \alpha = 0.3$ and $A\sigma_0 = 0.15$ MPa. Different lines correspond to different values of $\mu$. It is clear that $\mu = \tau_0/\sigma_0 - \alpha$ produces the least error.

$$K(t) = \exp\left(\frac{\tau(t)}{A\sigma(t)} - \frac{\tau_0}{A\sigma_0}\right)\left(\frac{\sigma(t)}{\sigma_0}\right)^{\alpha/A} \simeq \exp\left(\frac{\Delta\tau(t) - (\tau_0/\sigma_0 - \alpha)\Delta\sigma(t)}{A\sigma_0}\right), \tag{29}$$

where $\Delta\tau(t)$ and $\Delta\sigma(t)$ are the time-dependent changes in stress. The approximation is valid if

$$\left|\frac{\Delta\tau(t)\Delta\sigma(t)}{A\sigma_0^2}\right| \ll 1, \left|\frac{\alpha\Delta\sigma(t)^2}{2A\sigma_0^2}\right| \ll 1 \text{and} \left|\frac{\tau_0\Delta\sigma(t)^2}{A\sigma_0^3}\right| \ll 1, \tag{30}$$

at all times. If the normal stress is constant, all the error terms vanish and (29) is exact.

Implementing the approximation in equation (29) is equivalent to simplifying equation (2) and writing the stress dependence in terms of Coulomb stress change, $\Delta S(t) = \Delta\tau(t) - \mu\Delta\sigma(t)$, where $\mu = \tau_0/\sigma_0 - \alpha$, as before. This approach is currently more frequently applied than solving the state variable rigorously for large changes in normal stress as, for example, is shown in Appendix A or was done by Rubin and Ampuero (2007).

Figures 5a and 5b show that, given assumed values for various parameters, that choosing $\mu = \tau_0/\sigma_0 - \alpha$, leads to an accurate approximation. This is not surprising because in this example, $|\Delta\sigma/\sigma_0| \simeq 0.002$, and at $t = 10t_a$, $|\dot{\sigma}_r t/\sigma_0| \simeq 0.02$. Thus, approximating both the background stressing and the arbitrary stress kernel $K$ in terms of Coulomb stress change is reasonable, as long as the changes in normal stress are modest. However, choosing an incorrect value to $\mu$ may result in significant errors. Since the value of $\mu$ is generally not known a priori one could estimate it as a free parameter (within some bounds). Otherwise, it might bias the estimation of other parameters such as $A\sigma_0$.

To test how well the Coulomb stress approximations hold up in the presence of relatively large changes in normal stress, we repeat simulations in Figures 5a and 5b but increase the $\Delta\sigma$ and $\dot{\sigma}_r$ by factor of 10 while keeping all other parameters constant (Figures 5c and 5d). Thus now $|\Delta\sigma/\sigma_0| \simeq 0.02$, and at $t = 10t_a$, $|\dot{\sigma}_r t/\sigma_0| \simeq 0.2$. For the correct value of $\mu$ the error is typically $\lessapprox 10\%$, which would generally be acceptable, however, for incorrect values of $\mu$ the error can be very large. Figures 5c and 5d suggests that the approximations made in this section can be reasonable even in regions where normal stressing plays a dominant role in earthquake triggering and tectonic stressing if $\mu$ is selected or estimated carefully.

The Coulomb stress change approximation leads to a complete tradeoff between $A$ and $\sigma_0$, but the general Kernel in (10) and (29) suggests that there is a possibility that $A$ and $\sigma_0$ could be determined independently, at least to some degree. If the inequalities in (30) are significant then using (10) is not only more accurate but may also help to determine if $A$ is truly much smaller than laboratory experiments suggest, assuming $\sigma_0$ is the difference in lithostatic and hydrostatic pressure (Gomberg et al., 2000; Hainzl et al., 2010), or alternatively, the effective normal stress is smaller than expected. It should be noted that the Coulomb stress change approximation can be applied to all previous equations by substituting $K(t) = \exp(\Delta S(t)/A\sigma_0)$.

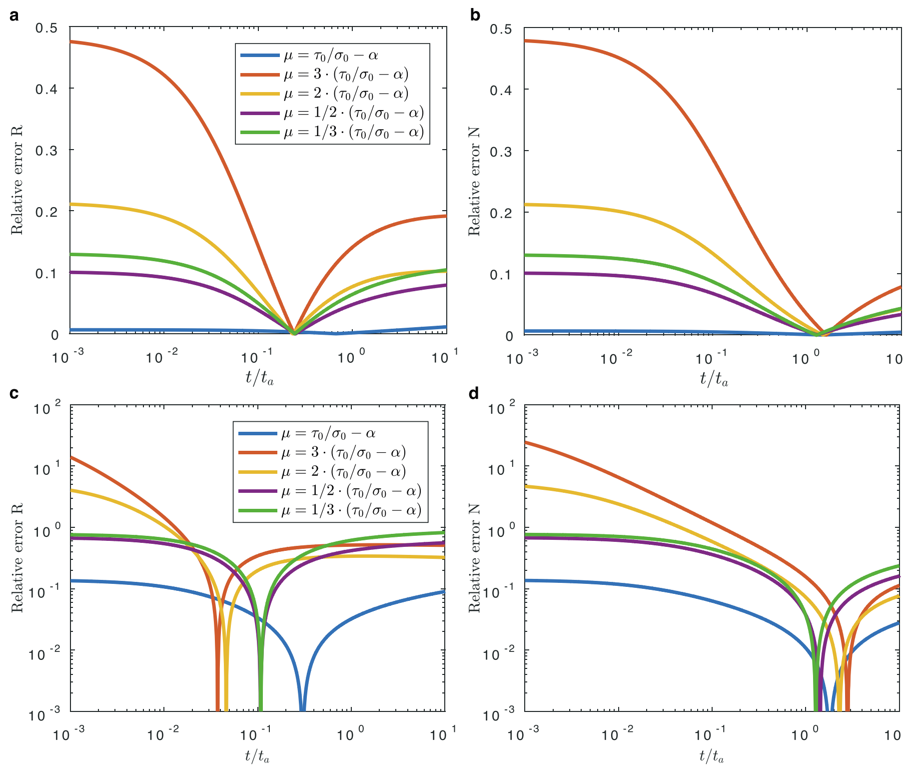


**Figure 5.** (a and b) Relative error in Coulomb stress approximation, as in Figure 4, however, here the Coulomb stress perturbation $K = \exp(\Delta S(t)/A\sigma_0)$ is made for the kernel in (28), as well as for the background stressing conditions. All parameters are the same as in Figure 4. (c and d) The same as (a) and (b) except with 10 larger value for $\Delta\sigma$ and $\dot{\sigma}_r$. Note that relative error for (c) and (d) is shown on log scale.

### 3.3. Nonconstant Background Seismicity Rate

The assumption that background seismicity rate $r$ is constant begs the questions of how the constitutive relationships will be changed if that is not the case, here we explore how to relax this assumption. For example, where stress shadows have occurred in the past there may be no measurable seismicity rate until the tectonic stress reaches some threshold, after which a constant rate of seismicity is produced, but how would the seismicity rate respond to an arbitrary stressing history?

Under the assumption of constant background shear stressing rate, (18) and (20) can be generalized for any function $\mathcal{B}$ that relates the background distribution of times to instability to the number of events, that is, $t = \mathcal{B}(N)$. In the case of constant background rate $r$ then $\mathcal{B}(N) = N/r$. Substituting $\mathcal{B}(N)$ for $N/r$ in (17),

$$\int_0^t K(t')\mathrm{d}t' = \int_0^{\mathcal{B}(N)} \exp(\dot{\tau}_r t'/A\sigma_0)\mathrm{d}t', \tag{31}$$

leads to a general relationship between the cumulative number of events assuming constant background shear stressing conditions:

$$N = \mathcal{B}^{-1}(N_r), \tag{32}$$

where $N_r = N/r$, as in equation (18). Taking the time derivative gives the seismicity rate

$$R = \frac{\partial N}{\partial t} = \frac{\partial \mathcal{B}^{-1}(N_r)}{\partial N_r}\frac{\partial N_r}{\partial t} = \frac{\partial \mathcal{B}^{-1}(N_r)}{\partial N_r}R_r, \tag{33}$$

where $R_r = R/r$, as in equation (20).

If one were to introduce a threshold such that no earthquakes nucleate before $N_r$ reaches a critical value, but after that produces a constant seismicity rate $r$ under background conditions, then $\mathcal{B}^{-1}$ would be a ramp

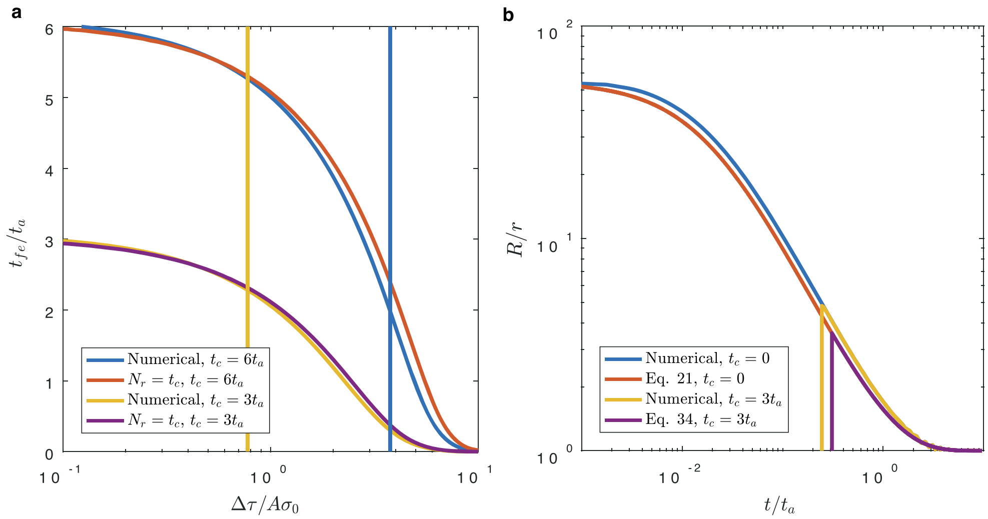


**Figure 6.** (a) Time of first event $t_{fe}$ for different stress steps. Numerical results based on section 3.1 are compared to prediction from (34) for two different values of $t_c$, the time to first event under background stressing: $t_c = 6t_a$ and $3t_a$. The first source to fail is moved above steady state instantly by the stress step for $\Delta\tau/A\sigma_0 \geq 3.7$ for $t_c = 6t_a$ (blue vertical line) and $\Delta\tau/A\sigma_0 \geq 0.8$ for $t_c = 3t_a$ (yellow vertical line). (b) Normalized seismicity rate, $R/r$. Comparison of numerical results to (34) for $t_c = 0$ (where equation (34) reduces to (21)) and $t_c = 3t_a$ for $\Delta\tau/A\sigma_0 = 4$. For $t_c = 3t_a$ all sources are below steady state prior to perturbation and thus the seismicity rate is zero for $t \leq t_{fe}$.

function $\mathcal{B}^{-1}(N_r) = r\mathcal{R}(N_r - t_c)$ and the derivative the Heaviside step function $r\mathcal{H}(N_r - t_c)$, where $t_c$ is the time of first event under under background stressing conditions, which could be linked to a critical stress threshold $\tau_c$ through the following equation $t_c = (\tau_c - \tau_0)/\dot{\tau}_r$.

It is easy see that once $N_r$ reaches the critical value at time $t_c$ then the predicted rate would be identical to that in (20), and thus, the cumulative number of events with reference to the time of first event would be the same. That is, the cumulative number of events and rate are

$$\begin{aligned} N/r &= \mathcal{R}(N_r - t_c) \\ R/r &= \mathcal{H}(N_r - t_c)R_r. \end{aligned} \tag{34}$$

Equation (34) suggest that the time of first event $t_{fe}$ is related to the time of the first event under background loading through $N_r(t = t_{fe}) = t_c$. Thus, $t_c - t_{fe}$ can be thought of as the *clock advance* of the seismicity. However, these derivations are based on the assumption that sources are well above steady state. It is worth considering how well the condition $N_r(t = t_{fe}) = t_c$ predicts $t_{fe}$ when the first source is initially not well above steady state. Figure 6a shows $t_{fe}$ compared to numerical simulations from section 3.1 for $t_c = 6t_a$ and $3t_a$. All sources are initially well below steady state prior to the stress step. The vertical lines mark the magnitude of the stress steps below which all sources are below steady state even after the stress step. The comparison indicates that equation (34) predict the time of the first event reasonably accurately, even when sources are not brought above steady state by the stress step. Figure 6b shows the seismicity rate as a function of normalized time for $t_c = 3t_a$ and $t_c = 0$ (where for the latter (34) reduces to (21). For $t_c = 3t_a$ the time to the first earthquake, $t_{fe}$, can be seen from Figure 6a to be $\sim 0.2t_a$; thus, the seismicity rate is zero for earlier times. Figure 6b shows that the seismicity rate predicted assuming the sources to be well above steady state agrees reasonably well with numerical simulations that include healing, even when all sources are initially below steady state.

A simple interpretation of the Dieterich model might suggest that if a particular area has no measurable background rate of seismicity then there must be no triggered seismicity, because if $r = 0$ then $R = 0$. We have shown here that this is not necessarily the case if the background stressing has not yet been sufficient to drive the seismogenic populations to a state of steady earthquake production. Equation (34) may be appropriate for regions that are seismically quiet but produces earthquakes when subject to larger stresses, for example, in some volcanic or intraplate settings or alternatively areas and faults that have been affected by significant stress shadows or stress drops.

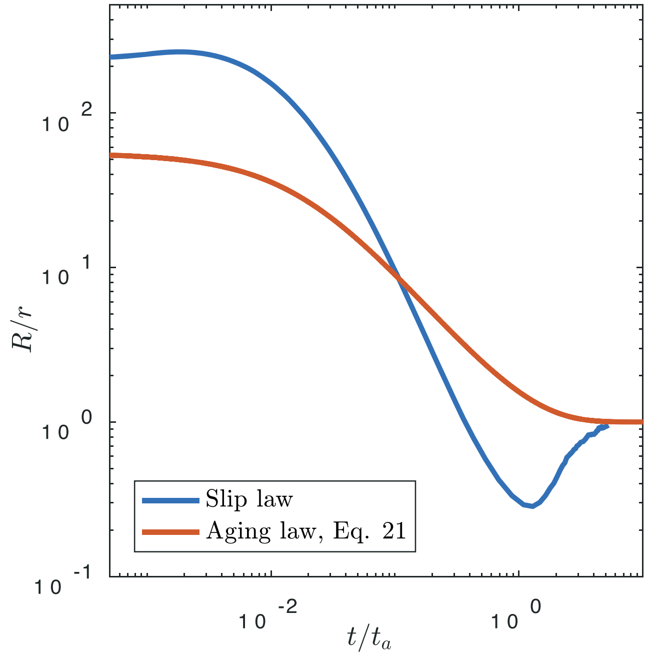


**Figure 7.** Comparison of the seismicity rate following a stress step $\Delta\tau/A\sigma_0 = 4$ as predicted by the slip law and aging law (equation (21)). Slip law predicts a decay of seismicity rate that is faster than observed in nature.

## 4. Discussion

### 4.1. Validity of the Dieterich Model

In this study we have rederived and extended the Dieterich model. Although the original model makes many assumptions, some of which might seem unreasonable, we have found that in general that it holds up fairly well to further inspection. For example, it was not explicitly shown to be valid for an arbitrary stressing history. Here we have shown that it is valid for arbitrary shear stressing history if the normal stress is constant but found a slightly different expression if the normal stress is time varying. The difference in the two expressions is unlikely to be significant unless the normal stress change is large.

There are number of questions concerning state evolution and its effects on seismicity rate that have not been fully addressed. The assumption that sources are well above steady state, equation (7), is not always appropriate. In section 3.1 we showed that sources that participate in an aftershock sequence may be initially below steady state but are elevated above steady state by the step change in stress. Although this may be valid for a rapid stress step, it is still not fully known what the consequences are for more gradual stress changes, or for stress shadows when healing may become important. Nevertheless, the Dieterich model has been applied to investigate stress shadows (e.g., Maccaferri et al., 2013). Another question that should be addressed in future work, is if the "slip law" for state evolution (Ruina, 1983) rather than the "aging law" (as used here) will lead to a significantly different relationship between stress and seismicity rate. Gomberg et al. (2000) suggested, based on simulations, that the slip law was not consistent with $1/t$ decay of aftershocks. Our simulations (Figure 7) agree with Gomberg et al. (2000) and suggest that if state evolves according to the slip law for a population of sources that fail at constant rate under constant background stressing rate, then the decay of seismicity rate is faster than observed in aftershock sequences (at least for some choices of initial conditions). Fully analyzing what gives rise to these differences is difficult without a comparable analytical solution based on the slip law.

Perhaps the most common criticism of the Dieterich model is that it neglects interaction between sources. However, as was shown by Ziv and Rubin (2003) this may not be as serious as it seems at first glance, at least for the decay in seismicity rate following a stress perturbation. This is because interactions affect both the effective background stressing rate as well as stress transfers, which counteract one another causing the characteristic aftershock decay time to scale with $A\sigma_0/\dot{\tau}_r$ as is the case without interactions.

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### 4.2. Is $\tau_0$ Constant in the Population?

Assuming that $\tau_0$ is independent of the source number, $N$ is generally not physically plausible, since individual sources may be at different stages of their seismic cycle. However, this assumption is necessary to attain simple explicit solutions for $R$, equation (20), when normal stress is not constant. In section 3.1 we noted that sources at different initial slip speeds may be at different initial stresses. This raises the question if it is reasonable to consider $\tau_0$ as constant in its contributions to the kernel $K(t)$, but not in the initial slip speed? Clearly if normal stress is constant then $K = \exp(\Delta\tau(t)/A\sigma_0)$ and there is no dependence on initial shear stress. Equation (29) provides insight into the effect of variable $\tau_0$ in presence of normal stress change. Denote $\tau_0 = \bar{\tau}_0 + d\tau$ where $\bar{\tau}_0$ is the mean initial shear stress among the population of sources, and $d\tau$ the maximum deviation from the mean. Inserting $\tau_0 = \bar{\tau}_0 + d\tau$ into (28) with $K$ approximated with equation (29) shows that change in the initial shear stress $d\tau$ can be ignored if

$$\left|\frac{d\tau}{\sigma_0}\frac{\Delta\sigma(t)}{\Delta S(t)}\right| \ll 1 \text{and} \left|\frac{d\tau}{\sigma_0}\frac{\dot{\sigma}_r}{\dot{s}_r}\right| \ll 1, \tag{35}$$

where $\mu = \bar{\tau}_0/\sigma_0$ in $\Delta S(t)$ and $\dot{s}_r$. Assuming that $\Delta\sigma(t)/\Delta S(t)$ and $\dot{\sigma}_r/\dot{s}_r$ are order unity, then $d\tau$ can be ignored if $|d\tau/\sigma_0| \ll 1$. If $d\tau$ is of the order of typical stress drops, then we would anticipate that the ratio $|d\tau/\sigma_0|$ is indeed small, suggesting that may be appropriate to interpret $\tau_0$ in $K$ is as the average initial shear stress. If conditions are such that inequalities (35) are violated then equations of the form of (26) are still valid, but some assumption about the distribution of initial stresses is required.

Section 3.1 (Figure 2b) suggests that one end member case is $\tau_0(N) = \tau_{max} - \Delta\tau_0 N$, where $\tau_{max}$ is the initial shear stress of source $N = 0$ and $\Delta\tau_0$ is the difference in initial stress between two consecutive sources. Another end member is that $\tau_0$ is constant but the differences in initial slip speeds arise from variations in state $\theta_0$. We numerically tested the case where $\theta_0(N)$ were chosen to produce a constant rate of seismicity under constant shear stressing with $\tau_0$ constant and found that a step change in shear stress also produced an Omori decay. We suspect, but did not test, that intermediate cases produce similar behavior and that $1/t$ decay involves sources either initially above steady state or brought there by the stress perturbation.

### 4.3. Generalized Constitutive Relationship

For practical applications, we favor equations (18) and (20) with or without the Coulomb stress approximation (equation (29)). However, it is worth presenting a more generalized from of the seismic productivity. In this work we showed that for a population of $N$ numbered seismic sources the time to instability $t$ of each source is given by the following equation, which summarizes most theoretical results of this study:

$$\int_0^t K_a(t', N(t))dt' = \int_0^{\mathcal{B}(N)} K_b(t', N(t))dt', \tag{36}$$

where $K_a$ is a general kernel that accounts for arbitrary stressing history and $K_b$ is a specific kernel that accounts for background stressing conditions. $\mathcal{B}(N)$ relates the time to instability and the source number under background stressing conditions. For constant background seismicity rate $r$, then $\mathcal{B}(N) = N/r$. The kernel functions may be functions of the source number $N$ if the initial conditions are not constant for all sources in the population, for example if the initial stress of the $N$-th source is $\tau_0(N)$. Equation 36 is general but not very practical, for example, there can be cases where the sources do not fail in the same order under background and general stressing conditions. In this case $N$ can no longer be interpreted as the cumulative number of events. In this study we investigated various different cases for $K_a$, $K_b$ or $\mathcal{B}(N)$; however, all of them can be written as special cases of (36).

## 5. Conclusions

The principal results of this paper, in terms of equations in the main text are as follows:

1. Equations (12) and (13) provide expressions for slip, $\delta$, and slip speed, $\dot{\delta}$, for arbitrary stressing history for a spring slider that is well above steady state.
2. Equation (14) gives a simple expression for the time to instability for a spring-slider subjected to arbitrary stressing history through the kernel $K(t)$, which could either be given by equation 10 or the Coulomb approximation equation 29.

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3. Equations (18) and (20) are the key results and give general expressions for the cumulative number of events $N(t)$, and seismicity rate $R(t)$, for arbitrary stressing history through the kernel $K(t)$. These expressions are considerably easier to use than the original Dieterich (1994) formulation.
4. Our results differ from Dieterich (1994), when the fault normal stress varies in time; however, in the limit of small changes in normal stress such that Coulomb approximation holds, the two formulations agree.

The new derivation presented in this paper allows one to generalize the Dieterich model and more readily test assumptions. We show how to include both shear and normal stressing under background conditions (e.g., equation (26)). We furthermore establish inequalities that should be met such that the kernel $K(t)$ can be simplified using the Coulomb stress approximation (equation (30)) and how assuming that a non-constant background seismicity rate affects predictions (equations 32 and 33). We found that the Omori $1/t$ behavior following a step change in stress, is produced by sources that are either initially above steady state, or are elevated above steady state by the stress step. This explains why the Dieterich model, which is based on the "no healing" (well above steady-state) approximation, is able to predict the ubiquitous Omori behavior.

## Appendix A: Integration of the State Variable

Dieterich (1994) derived the following ODE for the state variable

$$\dot{\gamma} = \frac{1}{A\sigma}\left[1 - \gamma\dot{\tau} + \gamma\left(\frac{\tau}{\sigma} - \alpha\right)\dot{\sigma}\right], \tag{A1}$$

where $A$ is a constitutive parameter that relates changes in friction and instantaneous slip velocity, $\sigma$ and $\tau$ are the effective normal stress and shear stress (respectively) acting on a seismogenic fault, and $\alpha$ is a constitutive parameter that relates change in normal stress to changes in state. Equation A1 is a first-order ODE with time-dependent coefficients and thus has a known semi-analytical solution (Polyanin & Zaitsev, 1995)

$$\gamma = e^{F(t)}\left(\frac{1}{\dot{\tau}_r} + \int_0^t e^{-F(t')}\frac{1}{A\sigma}\mathrm{d}t'\right) \tag{A2}$$

where function $F(t)$ is given by the following integral:

$$F(t) = \int_0^t \left[-\frac{\mathrm{d}\tau}{\mathrm{d}t'} + \left(\frac{\tau}{\sigma} - \alpha\right)\frac{\mathrm{d}\sigma}{\mathrm{d}t'}\right]\frac{1}{A\sigma}\mathrm{d}t'. \tag{A3}$$

Note that the solution in equations A2 and A can easily be verified by differentiating with respect to time and comparing to (A1). Equation (A3) can further be reduced to:

$$F(t) = -\int_0^t \frac{\mathrm{d}}{\mathrm{d}t'}\left(\frac{\tau}{A\sigma}\right)\mathrm{d}t' - \int_0^t \frac{\mathrm{d}}{\mathrm{d}t'}\left(\frac{\alpha}{A}\ln(\sigma)\right)\mathrm{d}t' \tag{A4}$$

The initial condition, $\gamma(t=0) = 1/\dot{\tau}_r$, requires $F(0) = 0$. Thus,

$$F(t) = -\left[\frac{\tau}{A\sigma} - \frac{\tau_0}{A\sigma_0} + \frac{\alpha}{A}\ln\left(\frac{\sigma}{\sigma_0}\right)\right] \tag{A5}$$

where the 0 subscript indicates the value at time $t = 0$. Thus, a semianalytical expression for $\gamma$ becomes

$$\gamma = \left(\frac{\sigma}{\sigma_0}\right)^{-\alpha/A}\exp\left(-\left[\frac{\tau}{A\sigma} - \frac{\tau_0}{A\sigma_0}\right]\right)\left[\frac{1}{\dot{\tau}_r} + \int_0^t \left(\frac{\sigma}{\sigma_0}\right)^{\alpha/A}\exp\left(\frac{\tau}{A\sigma} - \frac{\tau_0}{A\sigma_0}\right)\frac{1}{A\sigma}\mathrm{d}t'\right]. \tag{A6}$$

Inserting the expression above into into equation (1) gives:

$$R/r = \frac{\left(\frac{\sigma}{\sigma_0}\right)^{\alpha/A}\exp\left(\frac{\tau}{A\sigma} - \frac{\tau_0}{A\sigma_0}\right)}{1 + \dot{\tau}_r\int_0^t \left(\frac{\sigma}{\sigma_0}\right)^{\alpha/A}\exp\left(\frac{\tau}{A\sigma} - \frac{\tau_0}{A\sigma_0}\right)\frac{1}{A\sigma}\mathrm{d}t'}. \tag{A7}$$

Comparing this to equation 20, shows that they differ when normal stress $\sigma$ is time dependent. To yield a result consistent with (20) would require that

$$\gamma = e^{F(t)}\left(\frac{1}{\dot{\tau}_r} + \frac{1}{A\sigma_0}\int_0^t e^{-F(t')}\mathrm{d}t'\right). \tag{A8}$$

For the same $F(t)$, this yields the ODE,

$$\dot{\gamma} = \frac{1}{A\sigma_0} + \frac{1}{A\sigma(t)}\left[-\gamma\dot{\tau} + \gamma\left(\frac{\tau(t)}{\sigma(t)} - \alpha\right)\dot{\sigma}\right]. \tag{A9}$$

which can be compared to (A1).

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